# Sequentially Cohen-Macaulay Rees modules

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# §1 Introduction

**N. T. Cuong, S. Goto and H. L. Truong**

*The equality $I^2 = \mathfrak{q}I$ in sequentially Cohen-Macaulay rings*, J. Algebra, **(379)** (2013), 50-79.

In [CGT],

- Characterized the sequentially Cohen-Macaulay property of $\mathcal{R}(I)$ where $I$ is an $\mathfrak{m}$-primary ideal.

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When is the Rees module $\mathcal{R}(\mathcal{M})$ sequentially Cohen-Macaulay?

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# §2 Definition of sequentially C-M modules

Let $R$ be a Noetherian ring, $M \neq (0)$ a finitely generated $R$-module with $d = \dim_R M < \infty$. Then $\forall n \in \mathbb{Z}$,

$\exists M_n$ the largest $R$-submodule of $M$ with $\dim_R M_n \leq n$.

Let

$$\begin{aligned} \mathcal{S}(M) &= \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\} \\ &= \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M\} \\ &= \{d_1 < d_2 < \cdots < d_\ell = d\} \end{aligned}$$

where $\ell = \sharp \mathcal{S}(M)$.

Let $D_i = M_{d_i}$ for $1 \leq \forall i \leq \ell$. We then have a filtration

$$D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

which we call *the dimension filtration of $M$*. Put $C_i = D_i/D_{i-1}$ for $1 \leq \forall i \leq \ell$. Notice that $\dim_R D_i = \dim_R C_i = d_i$ for $1 \leq \forall i \leq \ell$.

### Definition 2.1 ([5, 6])

(1) $M$ is *a sequentially Cohen-Macaulay $R$-module*
$\overset{def}{\Longleftrightarrow}$ $C_i$ is a C-M $R$-module for $1 \leq \forall i \leq \ell$.

(2) $R$ is *a sequentially Cohen-Macaulay ring*
$\overset{def}{\Longleftrightarrow}$ $\dim R < \infty$ and $R$ is a sequentially C-M module over itself.

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$\overset{def}{\Longleftrightarrow}$ $\dim R < \infty$ and $R$ is a sequentially C-M module over itself.

# §3 Main results

In this section

- $(R, \mathfrak{m})$ a Noetherian local ring
- $M \neq (0)$ a finitely generated $R$-module with $d = \dim_R M$
- $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a filtration of ideals of $R$ s.t. $F_1 \neq R$
- $\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$ an $\mathcal{F}$-filtration of $R$-submodules of $M$
- $\mathcal{R} = \mathcal{R}(\mathcal{F})$ a Noetherian ring
- $\mathcal{R}(\mathcal{M})$ a finitely generated $\mathcal{R}$-module

Let $1 \leq i \leq \ell$. We set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n \in \mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n \in \mathbb{Z}}.$$

Then $\mathcal{D}_i$ (resp. $\mathcal{C}_i$) is an $\mathcal{F}$-filtration of $R$-submodules of $D_i$ (resp. $C_i$). We have the exact sequence

$$0 \to [\mathcal{D}_{i-1}]_n \to [\mathcal{D}_i]_n \to [\mathcal{C}_i]_n \to 0$$

of $R$-modules for $\forall n \in \mathbb{Z}$. Hence

$$0 \to \mathcal{R}(\mathcal{D}_{i-1}) \to \mathcal{R}(\mathcal{D}_i) \to \mathcal{R}(\mathcal{C}_i) \to 0$$

$$0 \to \mathcal{R}'(\mathcal{D}_{i-1}) \to \mathcal{R}'(\mathcal{D}_i) \to \mathcal{R}'(\mathcal{C}_i) \to 0 \quad \text{and}$$

$$0 \to \mathcal{G}(\mathcal{D}_{i-1}) \to \mathcal{G}(\mathcal{D}_i) \to \mathcal{G}(\mathcal{C}_i) \to 0.$$

## Theorem 3.1

*TFAE.*

(1) $\mathcal{R}'(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{G}$*-module and* $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \le i \le \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$.

*When this is the case,* $M$ *is a sequentially C-M* $R$*-module.*

## Theorem 3.2

*Suppose that* $M$ *is a sequentially C-M* $R$*-module and* $F_1 \not\subseteq \mathfrak{p}$ *for* $\forall \mathfrak{p} \in \mathrm{Ass}_R M$*. Then TFAE.*

(1) $\mathcal{R}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{G}$*-module,* $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \le i \le \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$ *and* $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ *for* $1 \le \forall i \le \ell$.

*When this is the case,* $\mathcal{R}'(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

# §4 Graded case

Let $R = \sum_{n \geq 0} R_n$ be a $\mathbb{Z}$-graded ring. We put

$$F_n = \sum_{k \geq n} R_k \quad \text{for} \quad \forall n \in \mathbb{Z}.$$

Then $F_n$ is a graded ideal of $R$, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration of ideals of $R$ and $F_1 := R_+ \neq R$.

Let $E$ be a graded $R$-module with $E_n = (0)$ for $\forall n < 0$. Put

$$E_{(n)} = \sum_{k \geq n} E_k \quad \text{for} \quad \forall n \in \mathbb{Z}.$$

Then $E_{(n)}$ is a graded $R$-submodule of $E$, $\mathcal{E} = \{E_{(n)}\}_{n \in \mathbb{Z}}$ is an $\mathcal{F}$-filtration of $R$-submodules of $E$.
Then we have

$$\underline{\underline{R = \mathcal{G}(\mathcal{F})}} \quad \text{and} \quad \underline{\underline{E = \mathcal{G}(\mathcal{E})}}.$$

## Assumption 4.1

- $R = \sum_{n \geq 0} R_n$ a Noetherian $\mathbb{Z}$-graded ring
- $E \neq (0)$ a finitely generated graded $R$-module with $d = \dim_R E < \infty$

## Proposition 4.2

*TFAE.*

(1) $\mathcal{R}'(\mathcal{E})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

(2) $E$ *is a sequentially C-M* $R$*-module.*

## Theorem 4.3

*Suppose that* $R_0$ *is a local ring,* $E$ *is a sequentially C-M* $R$*-module and* $F_1 \nsubseteq \mathfrak{p}$ *for* $\forall \mathfrak{p} \in \mathrm{Ass}_R E$*. Then TFAE.*

(1) $\mathcal{R}(\mathcal{E})$ *is a sequentially C-M* $\mathcal{R}$*-module.*

(2) $\mathrm{a}(C_i) < 0$ *for* $1 \leq \forall i \leq \ell$*.*

# §5 Application –Stanley-Reisner algebras–

## Notation 5.1

- $V = \{1, 2, \ldots, n\}$ $(n > 0)$ *a vertex set*
- $\Delta$ *a simplicial complex on* $V$ *s.t.* $\Delta \neq \emptyset$
- $\mathcal{F}(\Delta)$ *a set of facets of* $\Delta$
- $m = \sharp\mathcal{F}(\Delta)$ $(> 0)$ *its cardinality*
- $S = k[X_1, X_2, \ldots, X_n]$ *a polynomial ring over a field* $k$
- $I_\Delta = (X_{i_1} X_{i_2} \cdots X_{i_r} \mid \{i_1 < i_2 < \cdots < i_r\} \notin \Delta)$
- $R = k[\Delta] = S/I_\Delta$ *the Stanley-Reisner ring of* $\Delta$

We consider the $\mathbb{Z}$-graded ring $R = k[\Delta] = \sum_{n \geq 0} R_n$ and put

$$I_n := \sum_{k \geq n} R_k = \mathfrak{m}^n \quad \text{for} \quad \forall n \in \mathbb{Z}$$

where $\mathfrak{m} := R_+ = \sum_{n>0} R_n$. Then $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ is an $\mathfrak{m}$-adic filtration of $R$ and $I_1 \neq R$.

## Proposition 5.2

*If $\Delta$ is shellable, then $\mathcal{R}'(\mathfrak{m})$ is a sequentially C-M ring.*

Notice that

$$\mathfrak{p} \not\supseteq I_1 \text{ for } \forall \mathfrak{p} \in \operatorname{Ass} R \iff F \neq \emptyset \text{ for } \forall F \in \mathcal{F}(\Delta) \iff \Delta \neq \{\emptyset\}.$$

## Theorem 5.3

*Suppose that $\Delta$ is shellable with shelling order $F_1, F_2, \dots, F_m \in \mathcal{F}(\Delta)$ s.t. $\dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_m$ and $\Delta \neq \{\emptyset\}$. Then TFAE.*

(1) $\mathcal{R}(\mathfrak{m})$ *is a sequentially C-M ring.*

(2) $\dim F_i \geq \sharp \mathcal{F}(\langle F_1, F_2, \dots, F_{i-1} \rangle \cap \langle F_i \rangle)$ *for* $2 \leq \forall i \leq m$.

By Theorem 5.3 we get the following.

## Corollary 5.4

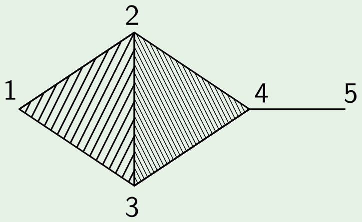
*Suppose that* $\dim F_m \geq 1$. *If* $\langle F_1, F_2, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ *is a simplex for* $2 \leq \forall i \leq m$, *then* $\mathcal{R}(\mathfrak{m})$ *is a sequentially C-M ring.*

## Example 5.5

Let $\Delta = \langle F_1, F_2, F_3 \rangle$, where $F_1 = \{1, 2, 3\}$, $F_2 = \{2, 3, 4\}$ and $F_3 = \{4.5\}$. Then $\Delta$ is shellable with numbering $\mathcal{F}(\Delta) = \{F_1, F_2, F_3\}$ and

$$\langle F_1 \rangle \cap \langle F_2 \rangle\,, \quad \langle F_1, F_2 \rangle \cap \langle F_3 \rangle$$

are simplexes, so that $\mathcal{R}(\mathfrak{m})$ is a sequentially C-M ring.

$\Delta =$ 

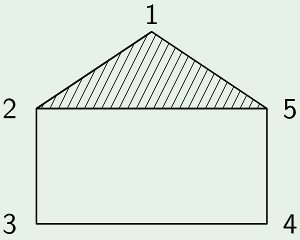
## Example 5.6

Let $\Delta = \langle F_1, F_2, F_3, F_4 \rangle$, where $F_1 = \{1, 2, 5\}$, $F_2 = \{2, 3\}$, $F_3 = \{3, 4\}$ and $F_4 = \{4, 5\}$. Then $\Delta$ is shellable with numbering $\mathcal{F}(\Delta) = \{F_1, F_2, F_3, F_4\}$. We put $\Delta_1 = \langle F_1, F_2, F_3 \rangle$, $\Delta_2 = \langle F_4 \rangle$. Then

$$\sharp\mathcal{F}(\Delta_1 \cap \Delta_2) = 2 = \dim F_4 + 1 > \dim F_4,$$

so that $\mathcal{R}(\mathfrak{m})$ is NOT a sequentially C-M ring by Theorem 5.3.

$\Delta =$



Thank you so much for your attention.

# References


[1] N. T. Cuong, S. Goto and H. L. Truong, *The equality $I^2 = \mathfrak{q}I$ in sequentially Cohen-Macaulay rings*, J. Algebra, **(379)** (2013), 50-79.

[2] G. Faltings, *Über die Annulatoren lokaler Kohomologiegruppen*, Archiv der Math., **30** (1978), 473–476.

[3] S. Goto, Y. Horiuchi and H. Sakurai, *Sequentially Cohen-Macaulayness versus parametric decomposition of powers of parameter ideals*, J. Comm. Algebra, **2** (2010), 37–54.

[4] S. Goto and K. Watanabe, *On graded rings, I*, J. Math. Soc. Japan, **30** (1978), 179–213.

[5] P. Schenzel, *On the dimension filtration and Cohen-Macaulay filtered modules*, in: Proc. of the Ferrara Meeting in honour of Mario Fiorentini, University of Antwerp, Wilrijk, Belgium, (1998), 245–264.

[6] R. P. Stanley, Combinatorics and commutative algebra, Second Edition, Birkhäuser, Boston, 1996.

[7] N. Taniguchi, T.T. Phuong, N. T. Dung and T. N. An, *Sequentially Cohen-Macaulay Rees modules*, preprint 2014.

[8] D. Q. Viet, *A note on the Cohen-Macaulayness of Rees Algebra of filtrations*, Comm. Algebra **21** (1993), 221-229.